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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Technical Memorandum 33-594

Analytic Expressions for Perturbations and Partial Derivatives of Range and Range Rate of a Spacecraft With Respect to the Coefficient of the Second Harmonic

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Prepared Under Contract No. NAS 7-100
National Aeronautics and Space Administration

PREFACE

The work described in this report was performed by the Mission Analysis Division of the Jet Propulsion Laboratory.

CONTENTS

ABSTRACT

Closed-form analytic expressions for the time variations of instantaneous orbital parameters and of the topocentric range and range rate of a spacecraft moving in the gravitational field of an oblate large body are derived using a first-order Variation of Parameters technique. In addition, the closed-form analytic expressions for the partial derivatives of the topocentric range and range rate are obtained, with respect to the coefficient of the second harmonic of the potential of the central body (J_2). The results are applied to the motion of a point-mass spacecraft moving in the orbit around the equatorially elliptic, oblate Sun, with $J_2 \cong 2.7 \times 10^{-5}$.

I. INTRODUCTION

The results of the theory of motion of a particle moving in a close orbit around a large oblate body are well known in the literature. Although the equation of the trajectory cannot be obtained in the closed form, a multitude of solutions to this problem have been found in many different forms, with different orders of accuracy, by means of power series of the coefficient of the second harmonic J_2.

The ultimate purpose of this study is twofold: first to determine the effects of the additional disturbing force generated by the oblateness of the central body on the observables — topocentric distance (range) and radial velocity (range rate) of a point-mass spacecraft, and secondly, to furnish the analytic expressions for partial derivatives of the two observables needed for the trajectory improvement analysis.

For close-to-Sun missions, assuming an oblate Sun having an oblateness coefficient $J_2 = 2.7 \times 10^{-5}$, due to the fact that the coefficient is very small, a first-order perturbation theory would yield sufficiently accurate results.

II. POTENTIAL FUNCTION OF THE CENTRAL BODY AND THE EQUATION OF MOTION OF THE SPACECRAFT

The potential of the attraction force exerted upon a particle by a large oblate body of mass M is given by (Refs. 1, 2)

$$V = -\frac{GM}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_0}{r}\right)^n P_n(\zeta)\right]$$

where r is the magnitude of the position vector of the particle relative to the center of mass of the large body, J_n are coefficients of zonal harmonics,

R_0 is the equatorial radius of the large central body, and GM is its gravitational constant; $P_n(\zeta)$ are Legendre polynomials of argument

$$\zeta = \sin\phi$$

where ϕ is the latitude of the particle measured from the equatorial plane of the central body.

Considering only the equatorial ellipticity of the central body, we can write the potential in the form

$$V = -\frac{GM}{r}\left[1 - J_2\left(\frac{R_0}{r}\right)^2 P_2(\zeta)\right] \tag{1}$$

where $P_2(\zeta)$ is the Legendre polynomial of the second order

$$P_2(\zeta) = \frac{1}{2}(3\zeta^2 - 1)$$

The acceleration of the attraction force, having a potential given by Eq. (1), is

$$\bar{a} = \frac{d\bar{v}}{dt} = -\text{grad } V = -\nabla V$$

Hence, the acceleration can be written in the form

$$\frac{d\bar{v}}{dt} = -\frac{GM}{r^3}\bar{r} + \bar{a}_P \tag{2}$$

where $\bar{a}_P$ is the additional, disturbing acceleration, caused by the equatorial ellipticity of the large body,

$$\bar{a}_P = -GM\,J_2 R_0^2 \nabla\left[\frac{P_2(\zeta)}{r^3}\right] \tag{3}$$

and $\bar{r}$ is the heliocentric position vector of the particle.

Let us introduce now an inertial frame of reference xyz, with the origin at the center of mass of the central body and the xy-plane coinciding with the equatorial plane of the central body. Then,

$$\sin\phi = \zeta = \frac{z}{r}, \qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

Performing the gradient operation indicated in Eq. (3), we find, for the disturbing part of the acceleration,

$$\bar{a}_P = \frac{3}{2} GMJ_2 \frac{R_0^2}{r^4}\left[(5\zeta^2 - 1)\frac{\bar{r}}{r} - 2\zeta\bar{k}\right]$$

As it is customary in the theory of orbital perturbations, we shall assume the solution of Eq. (2) in the quasi-elliptic form

$$r = \frac{p}{1 + e\cos\theta} \tag{4}$$

where the semi-latus rectum p and the eccentricity e are functions of time. Such an ellipse, different in shape and dimensions at each point, is known and referred to in the literature as the osculating ellipse, since the true trajectory of the particle and this ellipse have a common point and the same velocity at each instant during the motion of the particle (Refs. 3, 4). Using mathematical language, we can say that the true trajectory of the particle and the ellipse given by Eq. (4) have the tangency (or osculation) of the second order at every point during the motion. We must add here that the polar angle θ, also called the true anomaly of the particle, differs from the true anomaly of the unperturbed elliptic motion due to the action of the disturbing force.

Denoting by

$$\epsilon = \frac{3}{2} GMJ_2 \frac{R_0^2}{p^4} \tag{5}$$

we can write the expression for the disturbing acceleration in the form

$$\bar{a}_P = \epsilon\left(\frac{p}{r}\right)^4 \left[(5 \sin^2 \phi - 1)\frac{\bar{r}}{r} - 2\bar{k} \sin \phi\right] \qquad (6)$$

For the Sun, $J_2 = 2.7 \times 10^{-5}$ (Ref. 5). Using the values (Ref. 6)

$$GM = 1.327 \times 10^{11} \text{ km}^3/\text{s}^2$$

$$R_0 = 6.980 \times 10^5 \text{ km}$$

we find, with p given in astronomical units,

$$\epsilon = \frac{0.520}{p^4} \times 10^{-14} \text{ km/s}^2 \qquad (7)$$

Since $\epsilon \ll 1$, we can use the first-order Variation of Parameters method within the accuracy of $O(\epsilon^2)$. The parameters of the orbit (constants of integration of the Keplerian motion) which we shall use in this study are

a = semimajor axis of the osculating ellipse

e = its eccentricity

Ω = longitude of the ascending node, i.e., the angular distance between the positive direction of the x-axis and the line of nodes, in the equatorial plane of the central body

i = angle of inclination of the orbital plane to the equatorial plane of the central body

ω = angular distance between the direction of the ascending node and the periapsis, in the orbital plane. This angle is called the argument of periapsis.

The first two elements, a and e, define the dimension and shape of the osculating ellipse. The three angles, Ω, i, and ω, are the three Euler angles which describe the position in space of the orbital plane and the orientation of the osculating ellipse in that plane. These angles also determine the motion of the plane under the action of the disturbing force.

The equation of the Keplerian, undisturbed motion is a second-order vectorial differential equation, which, when integrated, contains six constants of integration. The choice of the sixth constant of integration, which is needed to introduce the time at which the particle is located at a certain position, is a matter of preference. In this study, we shall choose this sixth orbital parameter in the following manner. In Kepler's third law,

$$n^2 a^3 = GM = \mu \tag{8}$$

the quantity n represents the so-called mean motion, i.e., the constant angular velocity of a particle which, moving in a circle, has the same orbital period as the particle moving in an ellipse. The quantity

$$M(t) = n(t - T_0)$$

where T_0 is the time of the periapsis passage, is called the mean anomaly of the particle at time t, located at point P(t) (Fig. 1). Let us assume that, at a certain initial epoch t_0, we have an osculating ellipse, which we shall take as a reference orbit. According to the definition of the osculating orbit, if at time $t = t_0$ the perturbations stopped acting on the particle, it would continue to move in the initial reference ellipse and, at time t would be located at P*(t) (Fig. 1). The mean anomaly of the particle in that position is M*(t). We choose the sixth element τ to be the difference (Ref. 3)

$$\tau = M(t) - M^*(t) \tag{9}$$

The orbital geometry and Euler's angles Ω, i, and ω are shown in Fig. 2. From the spherical triangle shown in Fig. 3, we find

$$\zeta = \sin \phi = \sin i \sin (\theta + \omega) \tag{10}$$

Since our ultimate goal is to find the effect of the disturbing force on the topocentric range and range rate of the spacecraft, we have to find the coordinates of the spacecraft in an Earth equatorial system XYZ, where the XY-plane is the equatorial plane of the Earth. The geometry is shown in Fig. 4. If α and δ are the right ascension and declination of the north pole of the Sun, then, from Fig. 4,

$$\widehat{\Upsilon N} = \alpha - 270 \text{ deg}$$

$$i_N = 90 \text{ deg} - \delta$$

Taking the x-axis in the direction of the node N, we see that, in order to bring the coordinate system XYZ into the position of the system xyz, we have to perform the following two rotations:

(1) A positive rotation about the Z-axis by the angle $\widehat{\Upsilon N}$; this rotation brings the X-axis into the position of the x-axis;

(2) A positive rotation about the x-axis by the angle i_N. This rotation brings the Z-axis into the position of the z-axis.

Therefore, we can write the transformation equations

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \left[\widehat{\Upsilon N}\right]_z^T \left[i_N\right]_x^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{11}$$

and vice-versa,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[i_N\right]_x \left[\widehat{\Upsilon N}\right]_z \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{12}$$

where

$$\left[i_N\right]_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\delta & \cos\delta \\ 0 & -\cos\delta & \sin\delta \end{bmatrix} \tag{13}$$

and

$$
\left[\widehat{\gamma N} \right]_z = \begin{bmatrix} -\sin \alpha & \cos \alpha & 0 \\ -\cos \alpha & -\sin \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}
$$

With respect to the mean equinox and Earth equator of 1950.0, the spherical coordinates of the north pole of the Sun are (Ref. 7)

$$
\left.\begin{aligned} \alpha &= 286^\circ\!.0193 \\ \delta &= 63^\circ\!.7718 \end{aligned}\right\} \tag{15}
$$

The three Euler angles i, Ω, and ω of the orbital plane of the spacecraft, with respect to the equatorial plane of the central body (Sun), can be calculated from the corresponding angles i', Ω', and ω', given with respect to the equatorial plane of the Earth, in the following manner. From the spherical triangle NN_1N' (Fig. 4), we find

$$
\left.\begin{aligned} \cos i &= \sin \delta \cos i' - \cos \delta \sin i' \sin (\alpha - \Omega') \\ \sin i \cos (\omega' - \omega) &= \sin \delta \sin i' + \cos \delta \cos i' \sin (\alpha - \Omega') \\ \sin i \sin (\omega' - \omega) &= -\cos \delta \cos (\alpha - \Omega') \\ \sin i \cos \Omega &= -\cos \delta \cos i' - \sin \delta \sin i' \sin (\alpha - \Omega') \\ \sin i \sin \Omega &= -\sin i' \cos (\alpha - \Omega') \end{aligned}\right\} \tag{16}
$$

III. TWO INTEGRALS OF MOTION

The differential equation of motion (2), with the disturbing acceleration $\overline{a}_P$ given by Eq. (6), admits two integrals: the energy (vis-viva or Laplace's) integral and one scalar angular momentum integral. Indeed, writing the equation of motion in the form

$$\frac{d\bar{v}}{dt} = -\text{grad } V$$

and dot-multiplying both sides of this equation by $\bar{v}dt = d\bar{r}$, we find

$$\bar{v} \cdot d\bar{v} = -d\bar{r} \cdot \text{grad } V$$

and, since $d\bar{r} \cdot \text{grad } V = dV$, the integration of the above equation yields

$$\frac{v^2}{2} - \frac{v_0^2}{2} = -V + V(0)$$

or

$$v^2 = 2\frac{GM}{r} - \frac{4}{3}\epsilon p\left(\frac{p}{r}\right)^3 P_2(\zeta) + H \tag{17}$$

where the constant H has the value

$$H = v_0^2 + 2V(0) = v_0^2 - 2\frac{GM}{r_0} + \frac{4}{3}\epsilon p\left(\frac{p}{r_0}\right)^3 P_2(\zeta_0)$$

Since for the elliptic osculating motion at $t_0 = 0$,

$$v_0^2 = GM\left(\frac{2}{r_0} - \frac{1}{a(0)}\right)$$

the last equation for H transforms into

$$H = 2E_0 + \frac{4}{3}\epsilon p\left(\frac{p}{r_0}\right)^3 P_2(\zeta_0) \tag{18}$$

where

$$E_0 = -\frac{GM}{2a(0)}$$

is the total energy of the particle at $t_0 = 0$; $a(0)$ is the value of the semi-major axis at that instant. Also, $r_0 = r(0)$ and $\zeta_0 = \zeta(0)$.

To obtain the scalar angular momentum integral, we shall cross-multiply the differential equation (2) by vector $\overline{r}$ from the left. Thus, we obtain

$$\frac{d\overline{h}}{dt} = \overline{r} \times \overline{a}_P$$

where $\overline{h} = \overline{r} \times \overline{v}$ is the angular momentum vector. Substituting the value of $\overline{a}_P$ from Eq. (6), we obtain

$$\frac{d\overline{h}}{dt} = -2\epsilon\zeta\left(\frac{p}{r}\right)^4 (\overline{r} \times \overline{k}) \tag{19}$$

Vector $\overline{r} \times \overline{k}$ lies in the equatorial plane of the central body. Hence,

$$\frac{dh_z}{dt} = (\overline{r} \times \overline{k}) \cdot \overline{k} = 0$$

where h_z is the component of the vector $\overline{h}$ along the z-axis. Integrating the last equation, we have

$$h_z = \text{constant} = h_z(0)$$

From Fig. 2, we see that

$$h_z = \overline{h} \cdot \overline{k} = h \cos i$$

and the scalar angular momentum integral becomes

$$h \cos i = h_0 \cos i_0, \qquad i_0 = i(0),\ h_0 = h(0) \tag{20}$$

IV. PERTURBATIONS OF THE OSCULATING ORBITAL PARAMETERS

In order to obtain the time variations of the osculating orbital parameters due to the effect of the disturbing force, we shall first utilize the already obtained integrals of motion given by Eqs. (17) and (20). Since, for the osculating ellipse,

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{21}$$

we obtain by subtraction

$$\frac{1}{a(0)} - \frac{1}{a} = -2J_2R_0^2\left[\frac{P_2(\zeta)}{r^3} - \frac{P_2(\zeta_0)}{r_0^3}\right]$$

Further,

$$\frac{1}{a(0)} - \frac{1}{a} = \frac{a - a(0)}{a(0)a} = \frac{a - a(0)}{[a(0)]^2} + O\left(J_2^2\right)$$

and because of $J_2 \ll 1$, we can write

$$\Delta a = -\frac{4\epsilon}{3\mu}\left(a_0p_0\right)^2\left[\left(\frac{p}{r}\right)^3 P_2(\zeta)\right]_{t_0=0}^{t} \tag{22}$$

where $a_0 = a(0)$, $p_0 = p(0)$, and $\Delta a = a - a(0)$.

Taking the true anomaly as the variable, we obtain from Eq. (4)

$$\left(\frac{p}{r}\right)^3 = \left(1 + \frac{3e^2}{2}\right) + 3e\left(1 + \frac{e^2}{4}\right)\cos\theta + \frac{3e^2}{2}\cos 2\theta + \frac{e^3}{4}\cos 3\theta \tag{23}$$

Also, from

$$P_2(\zeta) = \frac{1}{2}\left[3\sin^2 i \sin^2(\theta + \omega) - 1\right]$$

we find

$$P_2(\zeta) = \frac{1}{4}\left[(3 \sin^2 i - 2) - 3 \sin^2 i \cos (2\theta + 2\omega)\right] \tag{24}$$

If we multiply Eqs. (23) and (24) and express everything in terms of multiple angles, we obtain, instead of Eq. (22), the following expression:

$$\begin{aligned}\Delta a = J_2K(a) \Bigg[&(2 - 3 \sin^2 i)\left\{(4 + e^2) \cos \theta + 2e \cos 2\theta + \frac{e^2}{3} \cos 3\theta\right\} \\ &+ \frac{\sin^2 i}{2} \Big\{ e^3 \cos (\theta - 2\omega) + 3e(4 + e^2) \cos (\theta + 2\omega) \\ &+ 4(2 + 3e^2) \cos (2\theta + 2\omega) + 3e(4 + e^2) \cos (3\theta + 2\omega) \\ &+ 6e^2 \cos (4\theta + 2\omega) + e^3 \cos (5\theta + 2\omega) \Big\}\Bigg]_{\theta_0}^{\theta}\end{aligned} \tag{25}$$

where $\theta_0 = \theta(0)$. The quantities e, i, and ω on the right-hand side of Eq. (25) should be written as e_0, i_0, and ω_0, but for the sake of simplicity, we shall drop the subscript zero in further writing. The constant K(a) is given by

$$K(a) = \frac{3}{8} \frac{\left(R_0 a\right)^2}{p^3} \tag{26}$$

where the subscript zero has been dropped again.

The second integral of motion, given by Eq. (20), yields

$$\frac{di}{dt} = \frac{\cot i}{h} \frac{dh}{dt} = \frac{\cot i}{h^2}\left(\bar{h} \cdot \frac{d\bar{h}}{dt}\right)$$

where $h = |\bar{h}|$. Substituting in this equation $d\bar{h}/dt$ given by Eq. 19), we find

$$\frac{di}{dt} = -2\epsilon\zeta \frac{\cot i}{h^2}\left(\frac{p}{r}\right)^4 (\bar{r} \times \bar{k}) \cdot \bar{h}$$

For the instantaneous osculating orbit,

$$r^2 \frac{d\theta}{dt} = h \tag{27}$$

so that, taking the true anomaly θ as the independent variable, we obtain

$$\frac{di}{d\theta} = -3J_2\left(\frac{R_0}{p}\right)^2 \cos i \left(\frac{p}{r}\right) \sin(\theta + \omega)\frac{d\zeta}{d\theta} \tag{28}$$

We shall now substitute p/r from Eq. (4) and

$$\frac{d\zeta}{d\theta} = \sin i \cos(\theta + \omega)$$

which we obtain by differentiating Eq. (10) with respect to θ. (For this differentiation, we keep i and ω constant.) After the substitution, Eq. (28) becomes

$$\frac{di}{d\theta} = -6J_2K(i)(1 + e\cos\theta)\sin(2\theta + 2\omega) \tag{29}$$

where

$$K(i) = \frac{1}{8}\left(\frac{R_0}{p}\right)^2 \sin 2i \tag{30}$$

Rewriting Eq. (29) in the form

$$\frac{di}{d\theta} = -3J_2K(i)\left[e\sin(\theta + 2\omega) + 2\sin(2\theta + 2\omega) + e\sin(3\theta + 2\omega)\right] \tag{31}$$

and integrating, keeping e, i, and ω fixed, we obtain

$$\Delta i = J_2 K(i) \Big[3e \cos(\theta + 2\omega) + 3 \cos(2\theta + 2\omega) + e \cos(3\theta + 2\omega) \Big]_{\theta_0}^{\theta} \tag{32}$$

where

$$\Delta i = i - i(0)$$

To find the time variation of the angular distance of the ascending node, Ω, we use the relationship (Ref. 3)

$$\frac{d\Omega}{d\theta} = \frac{\tan(\theta + \omega)}{\sin i} \frac{di}{d\theta}$$

Substituting $di/d\theta$ from Eq. (29), we obtain

$$\frac{d\Omega}{d\theta} = -6J_2 \frac{K(i)}{\sin i} \left[1 - \cos(2\theta + 2\omega) \right] (1 + e \cos \theta)$$

After the integration, we find

$$\Delta\Omega = -J_2 K(\Omega) \Big[6(\theta + e \sin\theta) - 3e \sin(\theta + 2\omega)$$

$$- 3 \sin(2\theta + 2\omega) - e \sin(3\theta + 2\omega) \Big]_{\theta_0}^{\theta} \tag{33}$$

where

$$K(\Omega) = \frac{1}{4} \left(\frac{R_0}{p} \right)^2 \cos i \tag{34}$$

and

$$\Delta\Omega = \Omega - \Omega(0)$$

As previously, the parameters e, i, and ω have been kept constant during this integration.

From Ref. 3, we further find

$$\frac{d\omega}{dt} + \frac{d\Omega}{dt} \cos i = - \frac{\delta\theta}{dt}$$

where δ/dt is the well known variational operator. Taking the angle θ as the independent variable, the last equation can be written in the form

$$\frac{d\omega}{d\theta} + \frac{d\Omega}{d\theta} \cos i = - \frac{\delta\theta}{d\theta} \tag{35}$$

Applying the variational operator to the polar equation of the osculating ellipse (4) and remembering that $\delta r = 0$* and that for every orbital parameter

$$\frac{\delta}{dt} \equiv \frac{d}{dt}$$

we find, taking the angle θ as the independent variable,

$$e \sin\theta \frac{\delta\theta}{d\theta} = \frac{de}{d\theta} \cos\theta - \frac{2p}{r} \tan i \frac{di}{d\theta} \tag{36}$$

Also, from the definition of the semi-latus rectum,

$$p = \frac{h^2}{\mu} = a(1 - e^2) \tag{37}$$

we find, by differentiation,

$$\frac{de}{d\theta} = \frac{p}{2a^2 e} \frac{da}{d\theta} - \frac{p}{ae} \tan i \frac{di}{d\theta} \tag{38}$$

*The Brown-Milankovic variational operator δ/dt is defined in the two basic equations of the Calculus of Perturbations

$$\begin{cases} \delta \bar{r} = 0 \\ \delta \bar{v} = \bar{a}_P dt \end{cases}$$

Considering a, e, and i constant, we obtain, by integration,

$$\Delta e = \frac{p}{ae}\left(\frac{\Delta a}{2a} - \tan i \, \Delta i\right) \tag{39}$$

Combining Eqs. (36) and (38), we find

$$e \sin\theta \frac{\delta\theta}{d\theta} = \frac{p \cos\theta}{2a^2 e} \frac{da}{d\theta} - \frac{2e + (1 + e^2)\cos\theta}{e} \tan i \frac{di}{d\theta} \tag{40}$$

From Eq. (25), we obtain, by differentiation,

$$\begin{aligned}
\frac{da}{d\theta} = -J_2 K(a) \Bigg[&(2 - 3 \sin^2 i)\left\{(4 + e^2)\sin\theta + 4e \sin 2\theta + e^2 \sin 3\theta\right\} \\
&+ \frac{\sin^2 i}{2}\Big\{e^3 \sin(\theta - 2\omega) + 3e(4 + e^2)\sin(\theta + 2\omega) \\
&\quad + 8(2 + 3e^2)\sin(2\theta + 2\omega) \\
&\quad + 9e(4 + e^2)\sin(3\theta + 2\omega) + 24e^2 \sin(4\theta + 2\omega) \\
&\quad + 5e^3 \sin(5\theta + 2\omega)\Big\}\Bigg]
\end{aligned} \tag{41}$$

Subsituting expressions (41) and (31) into Eq. (40) and dividing the result by $e \sin\theta$, we obtain

$$\begin{aligned}
\frac{\delta\theta}{d\theta} = -J_2 K(\omega) \Bigg[&(2 - 3 \sin^2 i)\left(1 + \frac{4 + 3e^2}{4e}\cos\theta + \cos 2\theta + \frac{e}{4}\cos 3\theta\right) \\
&+ \frac{\sin^2 i}{8}\Big\{e \cos(\theta - 2\omega) - \frac{4 - 7e^2}{e}\cos(\theta + 2\omega) \\
&\quad + 24 \cos(2\theta + 2\omega) + \frac{28 + 11e^2}{e}\cos(3\theta + 2\omega) \\
&\quad + 24 \cos(4\theta + 2\omega) + 5e \cos(5\theta + 2\omega)\Big\}\Bigg]
\end{aligned} \tag{42}$$

with

$$K(\omega) = \frac{3}{4}\left(\frac{R_0}{p}\right)^2 \tag{43}$$

An integration of Eq. (35) with the angle i considered as constant yields

$$\Delta\omega + \Delta\Omega \cos i = -\int_{\theta_0}^{\theta} \left(\frac{\delta\theta}{d\theta}\right) d\theta$$

or, after the performed integration on the right-hand side of the last equation,

$$\begin{aligned}\Delta\omega + \Delta\Omega \cos i = J_2 K(\omega) \Bigg[&(2 - 3 \sin^2 i)\left(\theta + \frac{4 + 3e^2}{4e} \sin\theta + \frac{1}{2} \sin 2\theta + \frac{e}{12} \sin 3\theta\right) \\ &+ \frac{\sin^2 i}{24} \Bigg\{ 3e \sin(\theta - 2\omega) - \frac{12 - 21e^2}{e} \sin(\theta + 2\omega) \\ &+ 36 \sin(2\theta + 2\omega) + \frac{28 + 11e^2}{e} \sin(3\theta + 2\omega) \\ &+ 18 \sin(4\theta + 2\omega) + 3e \sin(5\theta + 2\omega) \Bigg\} \Bigg]_{\theta_0}^{\theta}\end{aligned} \tag{44}$$

Incidentally, it is interesting to note at this stage that the expressions for Δa, Δe, Δi, $\Delta\Omega$, and $\Delta\omega$, given by Eqs. (25), (39), (32), (33), and (44), respectively, well agree with the known results. We find, namely, that Δa, Δe, and Δi are purely periodic, while $\Delta\Omega$ and $\Delta\omega$ contain secular terms. Denoting by

$$[\xi] = \frac{1}{2\pi}\int_0^{2\pi} \xi \, d\theta$$

the average value of an orbital element ξ over the period, we find that

$$\left.\begin{aligned} [\Delta a] &= 0, \\ [\Delta e] &= 0, \\ [\Delta i] &= 0, \\ [\Delta\Omega] &= -\frac{3}{2} J_2 \left(\frac{R_0}{p}\right)^2 \cos i, \\ [\Delta\omega] &= \frac{3}{4} J_2 \left(\frac{R_0}{p}\right)^2 (4 - 5 \sin^2 i) \end{aligned}\right\} \tag{45}$$

Hence, for

$$i = \arc \sin \left(\frac{2}{\sqrt{5}}\right) \cong 63^\circ.4$$

there is no secular advancement of the periapsis line.

According to what was said previously, the sixth orbital parameter, τ, is the difference between the perturbed and unperturbed mean anomaly at any instant t. Since (Ref. 3)

$$\frac{dM(t)}{dt} = n + \frac{\delta M(t)}{dt}$$

where δ/dt is the already used variational operator, and

$$\frac{dM^*(t)}{dt} = n_0$$

we find by subtraction (see Eq. 9) that

$$\frac{d\tau}{dt} = n - n_0 + \frac{\delta M(t)}{dt}$$

or

$$\frac{d\tau}{dt} = \Delta n + \frac{\delta M(t)}{dt} \tag{46}$$

Equation (46) can be written in a somewhat different form. From Kepler's third law (8), we obtain, by differentiation,

$$\Delta n = -\frac{3n}{2a}\Delta a$$

so that Eq. (46) takes the form

$$\frac{d\tau}{dt} = -\frac{3n}{2a}\Delta a + \frac{\delta M(t)}{dt} \tag{47}$$

Kepler's equation

$$\psi - e \sin \psi = M(t) \tag{48}$$

and the two relations which we derive from the Keplerian motion

$$r \cos \theta = a (\cos \psi - e)$$

$$r \sin \theta = b \sin \psi$$

where ψ is the so-called eccentric anomaly angle, and

$$b = a\sqrt{1 - e^2} \tag{49}$$

is the semiminor axis of the osculating ellipse, enable us to obtain $\delta M(t)/dt$ in terms of $\delta\theta/dt$ and de/dt. The expression which we find by using the operator δ/dt is

$$\frac{\delta M(t)}{dt} = \frac{r^2}{bp}\left[\frac{p}{a}\frac{\delta\theta}{dt} - \left(1 + \frac{p}{r}\right)\frac{de}{dt}\sin\theta\right]$$

or, taking again the true anomaly θ as the independent variable,

$$\frac{\delta M(t)}{d\theta} = \frac{r^2}{bp}\left[(1 - e^2)\frac{\delta\theta}{d\theta} - (2 + e\cos\theta)\sin\theta\,\frac{de}{d\theta}\right] \tag{50}$$

Combining Eqs. (36), (38), and (50), we find

$$\frac{\delta M(t)}{d\theta} = \frac{p}{abe^2\sin\theta}\left(\frac{p\cos\theta - 2er}{2a}\frac{da}{d\theta} - p\cos\theta\tan i\,\frac{di}{d\theta}\right) \tag{51}$$

and, substituting (da)/(dθ) and (di)/(dθ), respectively, from Eqs. (41) and (32), we obtain

$$\begin{aligned}\frac{\delta M(t)}{d\theta} = \frac{3J_2R_0^2}{4bpe}\Bigg[&(2 - 3\sin^2 i)\left\{e - \left(1 - \frac{5e^2}{4}\right)\cos\theta - e\cos 2\theta - \frac{e^2}{4}\cos 3\theta\right\}\\ &+ \frac{\sin^2 i}{8}\Big\{-e^2\cos(\theta - 2\omega) + (4 + 17e^2)\cos(\theta + 2\omega)\\ &+ 24e\cos(2\theta + 2\omega) - (28 - 13e^2)\cos(3\theta + 2\omega)\\ &- 24e\cos(4\theta + 2\omega) - 5e^2\cos(5\theta + 2\omega)\Big\}\Bigg]\end{aligned} \tag{52}$$

Although it is relatively simple to integrate Eq. (52), we shall do it by partially utilizing the already derived expression for $\Delta\omega + \Delta\Omega\cos i$ in Eq. (44). Namely, subtracting expression (42) multiplied by $\sqrt{1 - e^2}$ from Eq. (52), we find

$$\begin{aligned}\frac{\delta M(t)}{d\theta} - \sqrt{1 - e^2}\,\frac{\delta\theta}{d\theta} = J_2\sqrt{1 - e^2}\,K(\omega)\Big[&2(2 - 3\sin^2 i)(1 + e\cos\theta)\\ &+ 3\sin^2 i\,\big\{e\cos(\theta + 2\omega) + 2\cos(2\theta + 2\omega)\\ &+ e\cos(3\theta + 2\omega)\big\}\Big]\end{aligned} \tag{53}$$

Integrating the last equation, we obtain

$$\int_{\theta_0}^{\theta} \left(\frac{\delta M(t)}{d\theta}\right) d\theta + (\Delta\omega + \Delta\Omega \cos i)\sqrt{1 - e^2}$$

$$= J_2 \sqrt{1 - e^2}\, K(\omega) \Big[2(2 - 3 \sin^2 i)(\theta + e \sin \theta)$$

$$+ \sin^2 i \Big\{ 3e \sin (\theta + 2\omega) + 3 \sin (2\theta + 2\omega)$$

$$+ e \sin (3\theta + 2\omega) \Big\} \Big]_{\theta_0}^{\theta} \tag{54}$$

Equation (47) in the form (h = nab)

$$\frac{d\tau}{d\theta} = - \frac{3}{2a^2 b} (r^2 \Delta a) + \frac{\delta M(t)}{d\theta}$$

yields, after the integration,

$$\Delta\tau = - \frac{3}{2a^2 b} \int_{\theta_0}^{\theta} (r^2 \Delta a)\, d\theta + \int_{\theta_0}^{\theta} \left(\frac{\delta M(t)}{d\theta}\right) d\theta \tag{55}$$

where the last term on the right-hand side is given by Eq. (54).

From Eqs. (22) and (24),

$$r^2 \Delta a = \frac{J_2}{4p}\left(aR_0\right)^2 \Big[2(2 - 3\sin^2 i)(1 + e\cos\theta) + \sin^2 i\left\{3e\cos(\theta + 2\omega) + 6\cos(2\theta + 2\omega) + 3e\cos(3\theta + 2\omega)\right\}\Big] + 2J_2\left[\frac{\left(aR_0\right)^2}{r_0^3} P_2(\zeta_0)\right] r^2 \tag{56}$$

Hence, by integration,

$$\frac{3}{2a^2 b}\int_{\theta_0}^{\theta} (r^2 \Delta a)\, d\theta = 3J_2\left[\frac{naR_0^2}{r_0^3} P_2(\zeta_0)\right]\Delta t + \frac{J_2}{2}\sqrt{1 - e^2}\, K(\omega)\Big[2(2 - 3\sin^2 i)(\theta + e\sin\theta) + \sin^2 i\left\{3e\sin(\theta + 2\omega) + 3\sin(2\theta + 2\omega) + e\sin(3\theta + 2\omega)\right\}\Big]_{\theta_0}^{\theta} \tag{57}$$

Finally, substituting the results obtained in Eqs. (54) and (57) into Eq. (55), we find

$$\Delta\tau + (\Delta\omega + \Delta\Omega \cos i)\sqrt{1 - e^2} = -J_2 A \Delta t + J_2 \sqrt{1 - e^2}\, K(\omega)$$

$$\times \Bigg[(2 - 3 \sin^2 i)(\theta + e \sin \theta)$$

$$+ \frac{1}{2} \sin^2 i \Bigg\{ 3e \sin (\theta + 2\omega) + 3 \sin (2\theta + 2\omega)$$

$$+ e \sin (3\theta + 2\omega) \Bigg\} \Bigg]_{\theta_0}^{\theta} \tag{58}$$

where

$$A = 3 \frac{naR_0^2}{r_0^3} P_2(\zeta_0) \tag{59}$$

To find the change of τ over one period, we write, from Eq. (58),

$$[\Delta\tau] + \Big([\Delta\omega] + [\Delta\Omega] \cos i \Big) \sqrt{1 - e^2} = -J_2 A + J_2 (2 - 3 \sin^2 i) \sqrt{1 - e^2}\, K(\omega)$$

or, using the last two of Eqs. (45),

$$[\Delta\tau] = -J_2 A = -3J_2 \frac{naR_0^2}{r_0^3} P_2(\zeta_0) \tag{60}$$

From

$$P_2(\zeta_0) = \frac{1}{2}\Big[3 \sin^2 i \sin^2 (\theta_0 + \omega) - 1 \Big]$$

it follows that, within the accuracy of $O\left(J_2^2\right)$,

$$[\Delta\tau] = 0$$

if θ_0 is so chosen that

$$\sin\ (\theta_0 + \omega) = \pm \frac{1}{\sqrt{3}\ \sin i}$$

provided that

$$i \geq \text{arc sin}\left(\frac{1}{\sqrt{3}}\right) \cong 35^\circ.3$$

Equations (25), (32), (33), (39), (44), and (58) give the time variations of all six orbital parameters.

V. SUMMARY OF FORMULAE FOR Δa, Δe, Δi, $\Delta\Omega$, $\Delta\omega$, AND $\Delta\tau$

A. Auxiliary Quantities

We shall first compute the true anomaly angle θ for the time t using the equation of the unperturbed motion. In other words, assuming no change in orbital parameters, we compute the eccentric anomaly angle ψ at the time t from Eq. (48),

$$\psi - e \sin\psi = n(t - t_0) = n\Delta t$$

and the angle θ from

$$\tan\theta = \frac{\sqrt{1 - e^2}\ \sin\psi}{\cos\psi - e}$$

Next, we shall introduce the quantities

$$
\left.\begin{aligned}
C_N &= \Delta(\cos N\theta) = \cos(N\theta) - \cos(N\theta_0) \\
S_N &= \Delta(\sin N\theta) = \sin(N\theta) - \sin(N\theta_0)
\end{aligned}\right\} N = 1, 2, 3
$$

$$
K_0 = \Delta[\cos(\theta - 2\omega)] = \cos(\theta - 2\omega) - \cos(\theta_0 - 2\omega)
$$

$$
L_0 = \Delta[\sin(\theta - 2\omega)] = \sin(\theta - 2\omega) - \sin(\theta_0 - 2\omega) \qquad \text{Group [1]}
$$

$$
\left.\begin{aligned}
K_N &= \Delta[\cos(N\theta + 2\omega)] = \cos(N\theta + 2\omega) - \cos(N\theta_0 + 2\omega) \\
L_N &= \Delta[\sin(N\theta + 2\omega)] = \sin(N\theta + 2\omega) - \sin(N\theta_0 + 2\omega)
\end{aligned}\right\} N=1,2,\cdots,5
$$

$$
\Delta\theta = \theta - \theta_0
$$

The next step is to compute the constants

$$
\begin{aligned}
K(a) &= \frac{3}{8}\frac{(aR_0)^2}{p^3} \\
K(i) &= \frac{1}{8}\left(\frac{R_0}{p}\right)^2 \sin 2i \\
K(\Omega) &= \frac{1}{4}\left(\frac{R_0}{p}\right)^2 \cos i \\
K(\omega) &= \frac{3}{4}\left(\frac{R_0}{p}\right)^2 \qquad \text{Group [2]} \\
I_1 &= 2 - 3\sin^2 i \\
I_2 &= \sin^2 i \\
A &= 3\frac{naR_0^2}{r_0^3} P_2(\zeta_0)
\end{aligned}
$$

Having found all the necessary auxiliary quantities, we shall proceed with the calculation of the orbital parameter time variations.

B. Time Variations of Orbital Parameters

Substituting all the auxiliary quantities into Eqs. (25), (32), (33), (39), (44), and (58), we obtain

$$
\begin{aligned}
\Delta a &= J_2 K(a)\Bigg\{I_1\left[(4+e^2)C_1 + 2eC_2 + \frac{e^2}{3}C_3\right] \\
&\quad + \frac{1}{2}I_2\Big[e^3(K_0+K_5) + 3e(4+e^2)(K_1+K_3) \\
&\quad + 4(2+3e^2)K_2 + 6e^2K_4\Big]\Bigg\} \\
\Delta i &= J_2 K(i)(3eK_1 + 3K_2 + eK_3) \\
\Delta e &= \frac{1-e^2}{e}\left(\frac{\Delta a}{2a} - \tan i\ \Delta i\right) \\
\Delta\Omega &= -J_2 K(\Omega)\left(6\Delta\theta + 6eS_1 - 3eL_1 - 3L_2 - eL_3\right) \\
\Delta\omega &= -\Delta\Omega\cos i + J_2 K(\omega)\Bigg\{I_1\left(\Delta\theta + \frac{4+3e^2}{e}S_1\right. \\
&\quad \left. + \frac{1}{2}S_2 + \frac{e}{12}S_3\right) + \frac{1}{24}I_2\Bigg[3e(L_0+L_5) \\
&\quad - \frac{12-21e^2}{e}L_1 + \frac{28+11e^2}{e}L_3 \\
&\quad + 18\left(3L_2+L_4\right)\Bigg]\Bigg\}
\end{aligned}
\qquad \text{Group [3]}
$$

$$\Delta\tau = -(\Delta\omega + \Delta\Omega \cos i)\sqrt{1-e^2} - J_2 A\Delta t$$
$$+ J_2\sqrt{1-e^2}\,K(\omega)\Big[I_1(\Delta\theta + eL_1)$$
$$+ \frac{1}{2}I_2(3eL_1 + 3L_2 + eL_3)\Big] \qquad \text{Group [3]}$$

VI. PERTURBATIONS OF THE HELIOCENTRIC DISTANCE AND THE POLAR ANGLE OF THE SPACECRAFT

From Ref. 3, it follows that the disturbing effects on the heliocentric distance and the polar angle θ (the true anomaly) and their derivatives (radial and angular velocities, $\dot{r}$ and $\dot{\theta}$, respectively) are

$$\Delta r = \frac{r}{a}\Delta a - \cos\theta\Delta e + \frac{a^2 e}{b}\sin\theta\Delta\tau$$
$$\Delta\theta = \frac{a}{p}\left(1 + \frac{p}{r}\right)\sin\theta\Delta e + \frac{ab}{r^2}\Delta\tau$$
$$\Delta\dot{r} = -\frac{nae}{2b}\sin\theta\Delta a + nb\left(\frac{a}{r}\right)^2\sin\theta\Delta e$$
$$+ nae\left(\frac{a}{r}\right)^2\cos\theta\Delta\tau \qquad \text{Group [4]}$$
$$\Delta\dot{\theta} = -\frac{3nb}{2r^2}\Delta a + \frac{na^3}{br^2}\left(2\frac{p}{r}\cos\theta - e\right)\Delta e$$
$$- 2ne\left(\frac{a}{r}\right)^3\sin\theta\Delta\tau$$

In the first two equations of Group [4], Δr and $\Delta\theta$ are the differences between the heliocentric position and the polar angle of the particle in its perturbed trajectory at time t and the corresponding quantities related to an imaginary particle moving in the initial undisturbed elliptic orbit at the same time t. The same explanation holds for $\Delta\dot{r}$ and $\Delta\dot{\theta}$, given by the second two equations of Group [4].

VII. PARTIAL DERIVATIVES OF ORBITAL PARAMETERS WITH RESPECT TO J_2

Since the orbital parameters a_0, e_0, i_0, Ω_0, ω_0, and $M^*(t)$ of the initial osculating reference orbit do not depend on J_2, the partial derivative of any orbital parameter ξ with respect to J_2 is

$$\frac{\partial\xi}{\partial J_2} = \frac{\partial(\Delta\xi)}{\partial J_2}$$

Hence, we can write immediately

$$\frac{\partial(\Delta a)}{\partial J_2} = (DAJ) = K(a)\left\{ I_1\left[(4+e^2)C_1 + 2eC_2 + \frac{e^2}{3}(C_3)\right] + \frac{1}{2}I_2\left[e^3(K_0+K_5) + 3e(4+e^2)(K_1+K_3) + 4(2+3e^2)K_2 + 6e^2K_4\right]\right\}$$

$$\frac{\partial(\Delta i)}{\partial J_2} = (DIJ) = K(i)(3eK_1 + 3K_2 + eK_3)$$

$$\frac{\partial(\Delta e)}{\partial J_2} = (DEJ) = \frac{1-e^2}{e}\left[\frac{(DAJ)}{2a} - \tan i\,(DIJ)\right]$$

Group [5]

$$\frac{\partial(\Delta\Omega)}{\partial J_2} = (\text{DOJ}) = -K(\Omega)(6\Delta\theta + 6eS_1 - 3eL_1 - 3L_2 - eL_3)$$

$$\begin{aligned}\frac{\partial(\Delta\omega)}{\partial J_2} = (\text{DWJ}) = &-(\text{DOJ})\cos i - K(\omega)\Bigg\{I_1\left(\Delta\theta + \frac{4+3e^2}{e}S_1\right.\\ &\left.+\frac{1}{2}S_2 + \frac{e}{12}S_3\right) + \frac{1}{24}I_2\Bigg[3e(L_0 + L_5)\\ &-\frac{12-21e^2}{e}L_1 + \frac{28+11e^2}{e}L_3\\ &+18(3L_2 + L_4)\Bigg]\Bigg\}\end{aligned}$$

Group [5]

$$\begin{aligned}\frac{\partial(\Delta\tau)}{\partial J_2} = (\text{DTJ}) = &-\left[(\text{DWJ}) + (\text{DOJ})\cos i\right]\sqrt{1-e^2}\\ &-A\Delta t + K(\omega)\sqrt{1-e^2}\Big[I_1\left(\Delta\theta + eL_1\right)\\ &+\frac{1}{2}I_2(3eL_1 + 3L_2 + eL_3)\Big]\end{aligned}$$

VIII. CALCULATION OF ORBITAL PARAMETERS i, Ω, ω FROM EQUATORIAL ORBITAL PARAMETERS i', Ω', AND ω'

Most of the time in practice, the position of the orbital plane of a point-mass spacecraft, defined by the three Euler angles i, Ω, and ω, is given relative to one of the Earth reference planes: the equatorial or ecliptic plane for a certain fixed epoch, say 1950.0. The orbital elements i, Ω, and ω which we have used so far are relative to the equatorial plane of the central body, in particular, the equatorial plane of the Sun. To transfer from one system of Euler angles, i', Ω', ω' (Earth equatorial), which are normally given, to the other system of Euler angles, i, Ω, and ω, which we need for the calculation of perturbations of the position of the orbital plane in space, we have to use Eqs. (16). With α and δ given in Eqs. (15), the expressions from which we calculate i, Ω, and ω are

$$\boxed{\begin{aligned} \cos i &= \sin\delta \cos i' - \cos\delta \sin i' \sin(\alpha - \Omega') \\ \tan\Omega &= \frac{\sin i' \cos(\alpha - \Omega')}{\cos\delta \cos i' + \sin\delta \sin i' \sin(\alpha - \Omega')} \\ \tan(\omega - \omega') &= \frac{\cos\delta \cos(\alpha - \Omega')}{\sin\delta \sin i' + \cos\delta \cos i' \sin(\alpha - \Omega')} \end{aligned}} \qquad \text{Group [6]}$$

IX. PERTURBATIONS OF RANGE AND RANGE RATE

Let us denote by $\bar{\rho}$ the topocentric postion vector of the spacecraft relative to the observer's position at point P (Fig. 5), which is uniquely determined by the geocentric position vector $\bar{\xi}$. If $\bar{r}$ and $\bar{r}_E$ are the heliocentric position vectors of the spacecraft and the Earth, respectively, from the diagram shown in Fig. 5, we can write

$$\bar{\rho} = \bar{r} - \bar{r}_E - \bar{\xi} \tag{61}$$

Differentiating Eq. (61) with respect to time, we find

$$\dot{\bar{\rho}} = \dot{\bar{r}} - \dot{\bar{r}}_E - \dot{\bar{\xi}} \tag{62}$$

The disturbing effect of the equatorial ellipticity of the central body (Sun) affects not only the motion of the spacecraft but also the motion of the Earth. The position of the observer, however, defined by the geocentric position vector $\bar{\xi}$, remains unchanged, i.e., unaffected by the disturbing force.

Denoting by $\bar{r}^*$ and $\bar{r}^*_E$ the undisturbed position vectors of the spacecraft and the Earth, respectively, in other words the heliocentric position vectors of points in which the spacecraft and the Earth would be at time t if the perturbing effect were turned off at the time t_0, then, also

$$\bar{\rho}^* - \bar{r}^* - \bar{r}_E^* - \bar{\xi}$$

By subtraction of the last equation from Eq. (61), we obtain

$$\Delta\bar{\rho} = \bar{\rho} - \bar{\rho}^* = (\bar{r} - \bar{r}^*) - (\bar{r}_E - \bar{r}_E^*) = \Delta\bar{r} - \Delta\bar{r}_E \tag{63}$$

and, similarly,

$$\Delta\dot{\bar{\rho}} = \dot{\bar{\rho}} - \dot{\bar{\rho}}^* = (\dot{\bar{r}} - \dot{\bar{r}}^*) - (\dot{\bar{r}}_E - \dot{\bar{r}}_E^*)$$

$$= \Delta\dot{\bar{r}} - \Delta\dot{\bar{r}}_E$$

$$= \Delta\bar{v} - \Delta\bar{v}_E \tag{64}$$

where $\bar{v}$ and $\bar{v}_E$ are, respectively, the velocities of the spacecraft and the Earth.

Since

$$\bar{\rho} \cdot \Delta\bar{\rho} = \rho\Delta\rho$$

we find, dot-multiplying Eqs. (61) and (63), that

$$\Delta\rho = \frac{1}{\rho}\left[r\Delta r + r_E\Delta r_E - (\bar{r}_E + \bar{\xi}) \cdot \Delta\bar{r} - \bar{r} \cdot \Delta\bar{r}_E\right]$$

or, because $\xi/\rho << 1$,

$$\Delta\rho = \frac{1}{\rho}\left(r\Delta r + r_E\Delta r_E - \bar{r}_E \cdot \Delta\bar{r} - \bar{r} \cdot \Delta\bar{r}_E\right) \tag{65}$$

where, within the same order of accuracy,

$$\rho = \left[r^2 + r_E^2 + 2(\bar{r} \cdot \bar{r}_E)\right]^{1/2} \tag{66}$$

From

$$\bar{\rho} \cdot \dot{\bar{\rho}} = \rho\dot{\rho}$$

we obtain, by differentiation,

$$\Delta\dot{\rho} = \frac{1}{\rho}\left(\dot{\bar{\rho}} \cdot \Delta\bar{\rho} + \bar{\rho} \cdot \Delta\dot{\bar{\rho}} - \dot{\rho}\Delta\rho\right) \tag{67}$$

and, substituting expressions (63) and (64) into the last equation, we find, with the same accuracy as before,

$$\Delta\dot{\rho} = \frac{1}{\rho}\left[(\bar{v} - \bar{v}_E) \cdot (\Delta\bar{r} - \Delta\bar{r}_E) + (\bar{r} - \bar{r}_E) \cdot (\Delta\bar{v} - \Delta\bar{v}_E) - \dot{\rho}\Delta\rho\right] \tag{68}$$

where $\Delta\rho$ is given by Eq. (65).

Let X, Y, Z be the heliocentric rectangular coordinates of the spacecraft in the Earth equatorial space-fixed system of reference axes, and X_E, Y_E, Z_E the heliocentric equatorial coordinates of the Earth in the same system. In the xyz-reference frame, with the equatorial plane of the central body (Sun) as the fundamental plane, the coordinates of the spacecraft and the Earth and their velocity components are, respectively, (Fig. 4 and Eq. 12),

$$\boxed{\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \left[\delta\right]_x \left[\frac{\pi}{2} + \alpha\right]_z \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \\
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} &= \left[\delta\right]_x \left[\frac{\pi}{2} + \alpha\right]_z \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} \\
\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} &= \left[\delta\right]_x \left[\frac{\pi}{2} + \alpha\right]_z \begin{bmatrix} X_E \\ Y_E \\ Z_E \end{bmatrix} \\
\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} &= \left[\delta\right]_x \left[\frac{\pi}{2} + \alpha\right]_z \begin{bmatrix} \dot{X}_E \\ \dot{Y}_E \\ \dot{Z}_E \end{bmatrix}
\end{aligned}} \qquad \text{Group [7]}$$

The orbital plane parameters of the Earth, i_E, Ω_E, and ω_E, relative to the equatorial plane of the central body, can be easily determined in a manner similar to that used in determining the orbital plane elements of the spacecraft, using the equations of Group [6]. For the Earth, we should set $\Omega'_E = 0$, and $i'_E = \epsilon_E$, where ϵ_E is the obliquity of the ecliptic. In reality, however, the orbital plane of the Earth (using astronomers' jargon, we can call it true ecliptic, or instantaneous ecliptic plane) moves as a result of perturbations, and its position may be determined relative to the fixed Earth equatorial plane for the epoch 1950.0. From Ref. 7, we find, for ecliptic elements of the orbital plane of the Earth,

$$\left.\begin{aligned} i''_E &= 0^\circ\!.013076t - 0^\circ\!.000009\,t^2 \\ \Omega''_E &= 174^\circ\!.40956 - 0^\circ\!.24166t + 0^\circ\!.00006\,t^2 \\ \omega''_E &= 287^\circ\!.67097 + 0^\circ\!.56494t + 0^\circ\!.00009\,t^2 \end{aligned}\right\} \tag{69}$$

where the time is expressed in so-called Julian centuries, i.e., units of 36, 525 days. Similarly, the obliquity of the ecliptic is (Ref. 7)

$$\begin{aligned} i'_E = \epsilon_E(t) = {} & 23^\circ\!.4457888616 - 0^\circ\!.0130141669\,t \\ & - 0^\circ\!.09445 \times 10^{-5}\,t^2 + 0^\circ\!.05000 \times 10^{-5}\,t^3 \end{aligned} \tag{70}$$

The geometry of the orbital plane of the Earth is shown in Fig. 6. The orbital elements relative to the equatorial plane of the Earth for the epoch 1950.0 are obtained from the spherical triangle $\Upsilon N_1 N_2$. Equations corresponding to Eqs. (16) are

$$
\left.\begin{aligned}
\cos i'_E &= \cos i''_E \cos \epsilon_E - \sin i''_E \sin \epsilon_E \cos \Omega''_E \\
\sin i'_E \cos \Omega'_E &= \cos i''_E \sin \epsilon_E + \sin i''_E \cos \epsilon_E \cos \Omega''_E \\
\sin i'_E \sin \Omega'_E &= \sin i''_E \sin \Omega''_E \\
\sin i'_E \cos (\omega'_E - \omega''_E) &= \sin i''_E \cos \epsilon_E + \cos i''_E \sin \epsilon_E \cos \Omega''_E \\
\sin i'_E \sin (\omega'_E - \omega''_E) &= \sin \epsilon_E \sin \Omega''_E
\end{aligned}\right\} \quad (71)
$$

Thus, we obtain

$$
\boxed{\begin{aligned}
\cos i'_E &= \cos i''_E \cos \epsilon_E - \sin i''_E \sin \epsilon_E \cos \Omega''_E \\
\tan \Omega'_E &= \frac{\sin i''_E \sin \Omega''_E}{\cos i''_E \sin \epsilon_E + \sin i''_E \cos \epsilon_E \cos \Omega''_E} \\
\tan (\omega'_E - \omega''_E) &= \frac{\sin \epsilon_E \sin \Omega''_E}{\sin i''_E \cos \epsilon_E + \cos i''_E \sin \epsilon_E \cos \Omega''_E}
\end{aligned}} \quad \text{Group [8]}
$$

Taking now the line of nodes as the principal reference axis in the orbital plane of the spacecraft (Fig. 2), we see that the coordinates of the spacecraft in the plane are $(r \cos u, r \sin u, 0)$, where

$$u = \theta + \omega \quad (72)$$

The rectangular equatorial coordinates x, y, z can be derived from the coordinates in the orbital plane by two rotations: first, a positive rotation about the line of nodes by the inclination angle i; second, a negative rotation about the polar axis of the central body by the angle Ω. In other words,

$$
\left.\begin{aligned}
\begin{bmatrix} x \\ y \\ x \end{bmatrix} &= \begin{bmatrix} \Omega \end{bmatrix}_z^T \begin{bmatrix} i \end{bmatrix}_x^T \begin{bmatrix} r \cos u \\ r \sin u \\ 0 \end{bmatrix} \\[2ex]
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} \Omega \end{bmatrix}_z^T \begin{bmatrix} i \end{bmatrix}_x^T \begin{bmatrix} u \end{bmatrix}_z^T \begin{bmatrix} \dot{r} \\ r\dot{u} \\ 0 \end{bmatrix}
\end{aligned}\right\} \tag{73}
$$

For the Earth, we have two similar equations:

$$
\left.\begin{aligned}
\begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix} &= \begin{bmatrix} \Omega_E \end{bmatrix}_z^T \begin{bmatrix} i_E \end{bmatrix}_x^T \begin{bmatrix} r_E \cos u_E \\ r_E \sin u_E \\ 0 \end{bmatrix} \\[2ex]
\begin{bmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{bmatrix} &= \begin{bmatrix} \Omega_E \end{bmatrix}_z^T \begin{bmatrix} i_E \end{bmatrix}_x^T \begin{bmatrix} u_E \end{bmatrix}_z^T \begin{bmatrix} \dot{r}_E \\ r_E\dot{u}_E \\ 0 \end{bmatrix}
\end{aligned}\right\} \tag{74}
$$

Since the equations for the Earth are obtained just by adding the subscript E, we shall write only the equations for the spacecraft and keep in mind that the same equations exist for the Earth. In expressions (73) and (74),

$$
\begin{bmatrix} \Omega \end{bmatrix}_z^T, \begin{bmatrix} i \end{bmatrix}_x^T, \text{ and } \begin{bmatrix} u \end{bmatrix}_z^T
$$

are rotation matrices

$$
\left.\begin{aligned}
\left[\Omega\right]_z^T &= \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\left[i\right]_x^T &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \\
\left[u\right]_z^T &= \begin{bmatrix} \cos u & -\sin u & 0 \\ \sin u & \cos u & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}\right\} \qquad (75)
$$

Three rotation matrices can be written for the Earth by adding the subscript E to all quantities involved.

Let us now write the expressions for the disturbing effects in range and range rate of the spacecraft, $\Delta\rho$ and $\Delta\dot{\rho}$, given by Eqs. (65) and (68), respectively, in the following forms:

$$
\Delta\rho = \frac{1}{\rho}\begin{bmatrix} x - x_E & y - y_E & z - z_E \end{bmatrix} \begin{bmatrix} \Delta x - \Delta x_E \\ \Delta y - \Delta y_E \\ \Delta z - \Delta z_E \end{bmatrix} \qquad \text{Group [9]}
$$

$$\Delta \dot{\rho} = \frac{1}{\rho}\left\{\begin{bmatrix} \dot{x} - \dot{x}_E & \dot{y} - \dot{y}_E & \dot{z} - \dot{z}_E \end{bmatrix} \begin{bmatrix} \Delta_x - \Delta x_E \\ \Delta y - \Delta y_E \\ \Delta z - \Delta z_E \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} x - x_E & y - y_E & z - z_E \end{bmatrix} \begin{bmatrix} \Delta \dot{x} - \Delta \dot{x}_E \\ \Delta \dot{y} - \Delta \dot{y}_E \\ \Delta \dot{z} - \Delta \dot{z}_E \end{bmatrix} - \dot{\rho} \Delta \rho \right\}$$

$$\rho = \left[\left(x - x_E\right)^2 + \left(y - y_E\right)^2 + \left(z - z_E\right)^2 \right]^{1/2}$$

$$\dot{\rho} = \frac{1}{\rho}\begin{bmatrix} x - x_E & y - y_E & z - z_E \end{bmatrix} \begin{bmatrix} \dot{x} - \dot{x}_E \\ \dot{y} - \dot{y}_E \\ \dot{z} - \dot{z}_E \end{bmatrix}$$

Group [9]

From Eqs. (73) (and, subsequently, Eqs. 74), we find, by differentiation,

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \left\{ \left(\frac{\partial}{\partial \Omega} \begin{bmatrix} \Omega \end{bmatrix}_z^T \right) \begin{bmatrix} i \end{bmatrix}_x^T \Delta\Omega + \begin{bmatrix} \Omega \end{bmatrix}_z^T \left(\frac{\partial}{\partial i} \begin{bmatrix} i \end{bmatrix}_x^T \right) \Delta i \right\} \begin{bmatrix} r \cos u \\ r \sin u \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \Omega \end{bmatrix}_z^T \begin{bmatrix} i \end{bmatrix}_x^T \begin{bmatrix} u \end{bmatrix}_z^T \begin{bmatrix} \Delta r \\ r\Delta u \\ 0 \end{bmatrix} \qquad (76)$$

$$\begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{y} \\ \Delta\dot{z} \end{bmatrix} = \left\{ \left(\frac{\partial}{\partial \Omega} \left[\Omega \right]_z^T \right) \left[i \right]_x^T \left[u \right]_z^T \Delta\Omega \right.$$

$$+ \left[\Omega \right]_z^T \left(\frac{\partial}{\partial i} \left[i \right]_x^T \right) \left[u \right]_z^T \Delta i$$

$$\left. + \left[\Omega \right]_z^T \left[i \right]_x^T \left(\frac{\partial}{\partial u} \left[u \right]_z^T \right) \Delta u \right\} \begin{bmatrix} \dot{r} \\ r\dot{u} \\ 0 \end{bmatrix}$$

$$+ \left[\Omega \right]_z^T \left[i \right]_x^T \left[u \right]_z^T \begin{bmatrix} \Delta\dot{r} \\ \Delta(r\dot{u}) \\ 0 \end{bmatrix} \tag{77}$$

We have, however,

$$\left. \begin{aligned} \frac{\partial}{\partial \Omega} \left[\Omega \right]_z^T &= \left[\frac{3\pi}{2} - \Omega \right]_z \\ \frac{\partial}{\partial i} \left[i \right]_x^T &= \left[\frac{3\pi}{2} - i \right]_x \\ \frac{\partial}{\partial u} \left[u \right]_z^T &= \left[\frac{3\pi}{2} - u \right]_z \end{aligned} \right\} \tag{78}$$

so that, accordingly, Eqs. (76) and (77) become

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \left(\left[\frac{3\pi}{2} - \Omega\right]_z \left[i\right]_x^T \Delta\Omega + \left[\Omega\right]_z^T \left[\frac{3\pi}{2} - i\right]_x \Delta i \right) \begin{bmatrix} r \cos u \\ r \sin u \\ 0 \end{bmatrix}$$

$$+ \left[\Omega\right]_z^T \left[i\right]_x^T \left[u\right]_z^T \begin{bmatrix} \Delta r \\ r\Delta u \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \Delta \dot{x} \\ \Delta \dot{y} \\ \Delta \dot{z} \end{bmatrix} = \left(\left[\frac{3\pi}{2} - \Omega\right]_z \left[i\right]_x^T \left[u\right]_z^T \Delta\Omega + \left[\Omega\right]_z^T \left[\frac{3\pi}{2} - i\right]_x \left[u\right]_z^T \Delta i \right.$$

$$\left. + \left[\Omega\right]_z^T \left[i\right]_x^T \left[\frac{3\pi}{2} - u\right]_z \Delta u \right) \begin{bmatrix} \dot{r} \\ r\dot{u} \\ 0 \end{bmatrix} \qquad \text{Group[10]}$$

$$+ \left[\Omega\right]_z^T \left[i\right]_x^T \left[u\right]_z^T \begin{bmatrix} \Delta \dot{r} \\ r\Delta\dot{u} + \dot{u}\Delta r \\ 0 \end{bmatrix}$$

$$\dot{r} = \frac{he}{p} \sin\theta$$

$$\dot{u} = \frac{h}{r^2}$$

$$\Delta u = \Delta\theta + \Delta\omega$$

$$\Delta\dot{u} = \Delta\dot{\theta}$$

where Δr, $\Delta\dot{r}$, $\Delta\theta$, and $\Delta\dot{\theta}$ are given by the equations of Group [4].

X. PARTIAL DERIVATIVES OF RANGE AND RANGE RATE

The partial derivatives of the topocentric range and range rate can be derived directly from the equations for the perturbational changes $\Delta\rho$ and $\Delta\dot{\rho}$ by simply replacing the operator Δ with $\partial/\partial J_2$ in the equations of Groups [9] and [10]. Hence, from the equations of Group [10], we obtain the partial derivatives of the rectangular coordinates x, y, z and velocity components $\dot{x}, \dot{y}, \dot{z}$, with respect to J_2. Denoting by (Refs. 8 and 9)

$$\left.\begin{aligned}
\frac{\partial x}{\partial J_2} &= (DXJ) & \qquad \frac{\partial x_E}{\partial J_2} &= (DXEJ) \\
\frac{\partial y}{\partial J_2} &= (DYJ) & \qquad \frac{\partial y_E}{\partial J_2} &= (DYEJ) \\
\frac{\partial z}{\partial J_2} &= (DZJ) & \qquad \frac{\partial z_E}{\partial J_2} &= (DZEJ) \\
\frac{\partial \dot{x}}{\partial J_2} &= (DDXJ) & \qquad \frac{\partial \dot{x}_E}{\partial J_2} &= (DDXEJ) \\
\frac{\partial \dot{y}}{\partial J_2} &= (DDYJ) & \qquad \frac{\partial \dot{y}_E}{\partial J_2} &= (DDYEJ) \\
\frac{\partial \dot{z}}{\partial J_2} &= (DDZJ) & \qquad \frac{\partial \dot{z}_E}{\partial J_2} &= (DDZEJ)
\end{aligned}\right\} \qquad (79)$$

we can write

$$\begin{bmatrix} (DXJ) \\ (DYJ) \\ (DZJ) \end{bmatrix} = \left\{ \left[\frac{3\pi}{2} - \Omega\right]_z \left[i \right]_x^T (DOJ) \right.$$

$$\left. + \left[\Omega\right]_z^T \left[\frac{3\pi}{2} - i\right]_x (DIJ) \right\} \begin{bmatrix} r \cos u \\ r \sin u \\ 0 \end{bmatrix}$$

$$+ \left[\Omega\right]_z^T \left[i \right]_x^T \left[u \right]_z^T \begin{bmatrix} (DRJ) \\ r(DUJ) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (DDXJ) \\ (DDYJ) \\ (DDZJ) \end{bmatrix} = \left\{ \left[\frac{3\pi}{2} - \Omega\right]_z \left[i \right]_x^T \left[u \right]_z^T (DOJ) \right.$$

Group [11]

$$+ \left[\Omega\right]_z^T \left[\frac{3\pi}{2} - i\right]_x \left[u \right]_z^T (DIJ)$$

$$\left. + \left[\Omega\right]_z^T \left[i \right]_x^T \left[\frac{3\pi}{2} - u\right]_z (DUJ) \right\} \begin{bmatrix} \dot{r} \\ r\dot{u} \\ 0 \end{bmatrix}$$

$$+ \left[\Omega\right]_z^T \left[i \right]_x^T \left[u \right]_z^T \begin{bmatrix} (DDRJ) \\ r(DDUJ) + \dot{u}(DRJ) \\ 0 \end{bmatrix}$$

where

$$\left.\begin{aligned}
(DRJ) &= \frac{\partial r}{\partial J_2} \\
(DUJ) &= \frac{\partial u}{\partial J_2} \\
(DDRJ) &= \frac{\partial \dot{r}}{\partial J_2} \\
(DDUJ) &= \frac{\partial \dot{u}}{\partial J_2}
\end{aligned}\right\} \quad (80)$$

for the spacecraft, and

$$\left.\begin{aligned}
(DREJ) &= \frac{\partial r_E}{\partial J_2} \\
(DUEJ) &= \frac{\partial u_E}{\partial J_2} \\
(DDREJ) &= \frac{\partial \dot{r}_E}{\partial J_2} \\
(DDUEJ) &= \frac{\partial \dot{u}}{\partial J_2}
\end{aligned}\right\} \quad (81)$$

for the Earth.

Since r^* and θ^* do not depend on J_2, we find from the equations of Group [4] and Eq. (72) that

$$\boxed{\begin{aligned}
(DRJ) &= \frac{r}{a}(DAJ) - a\cos\theta\,(DEJ) + \frac{a^2 e}{b}\sin\theta\,(DTJ) \\
(DUJ) &= \frac{a}{p}\left(1+\frac{p}{r}\right)\sin\theta\,(DEJ) + \frac{ab}{r^2}(DTJ) + (DWJ) \\
(DDRJ) &= -\frac{nae}{2b}\sin\theta\,(DAJ) + nb\left(\frac{a}{r}\right)^2\sin\theta\,(DEJ) \\
&\quad + nae\left(\frac{a}{r}\right)^2\cos\theta\,(DTJ) \\
(DDUJ) &= -\frac{3nb}{2r^2}(DAJ) + \frac{na^3}{br^2}\left(2\frac{p}{r}\cos\theta - e\right)(DEJ) \\
&\quad - 2ne\left(\frac{a}{r}\right)^3\sin\theta\,(DTJ)
\end{aligned}} \qquad \text{Group [12]}$$

and similar expressions for the partial derivatives of r_E, $\dot{r}_E$, u_E, and $\dot{u}_E$, with respect to J_2.

Finally, we can write

$$\left.\begin{aligned}
\frac{\partial\rho}{\partial J_2} &= \frac{1}{\rho}\begin{bmatrix} x-x_E & y-y_E & z-z_E \end{bmatrix}\begin{bmatrix} (DXJ)-(DXEJ) \\ (DYJ)-(DYEJ) \\ (DZJ)-(DZEJ) \end{bmatrix} \\
\frac{\partial\dot{\rho}}{\partial J_2} &= \frac{1}{\rho}\left\{\begin{bmatrix} \dot{x}-\dot{x}_E & \dot{y}-\dot{y}_E & \dot{z}-\dot{z}_E \end{bmatrix}\begin{bmatrix} (DXJ)-(DXEJ) \\ (DYJ)-(DYEJ) \\ (DZJ)-(DZEJ) \end{bmatrix}\right. \\
&\quad + \begin{bmatrix} x-x_E & y-y_E & z-z_E \end{bmatrix}\begin{bmatrix} (DDXJ)-(DDXEJ) \\ (DDYJ)-(DDYEJ) \\ (DDZJ)-(DDZEJ) \end{bmatrix} - \dot{\rho}\frac{\partial\rho}{\partial J_2}
\end{aligned}\right\} \qquad (82)$$

The computational sequence in the form of a flow chart is shown in Fig. 7.

It should be mentioned, in conclusion, that a much higher accuracy is obtained if, instead of taking initial orbital parameters a_0, e_0, i_0, Ω_0, and ω_0 in the equations of Group [3] and thereafter, we repeat the calculations of Δa, Δe, Δi, $\Delta\Omega$, $\Delta\omega$, and $\Delta\tau$ with the mean values of elements

$$a_m = \frac{1}{2}(a_0 + a) \approx a_0 + \frac{\Delta a}{2}$$

$$e_m = \frac{1}{2}(e_0 + e) = e_0 + \frac{\Delta e}{2}$$

etc.

NOMENCLATURE

a	semimajor axis of the osculating ellipse at time t
$a_0 = a(0)$	semimajor axis of the initial osculating reference ellipse at time t_0
$\bar{a}$	acceleration vector of the particle
$\bar{a}_p$	disturbing acceleration vector
A	constant of integration
b	semiminor axis of the osculating ellipse
C_N	difference between cosines of angle $N\theta$ at time t and time t_0, for $N = 1, 2, 3$
e	eccentricity of the osculating ellipse
$e_0 = e(0)$	eccentricity of the initial osculating reference ellipse
E_0	total energy of the particle at time t_0
G	universal constant of gravitation
h	magnitude of the angular momentum vector
$h_0 = h(0)$	magnitude of the angular momentum vector at time t_0
h_z	component of the angular momentum vector along the z-axis

$h_z(0)$	component of the angular momentum vector along the z-axis at time t_0
$\bar{h}$	angular momentum vector
H	constant of integration
i	angle of inclination of the orbital plane of the spacecraft to the equatorial plane of the central body
$i_0 = i(0)$	angle of inclination of the orbital plane of the spacecraft to the equatorial plane of the central body at time t_0
i'	angle of inclination of the orbital plane of the spacecraft to the equatorial plane of the Earth for 1950.0
i_E	angle of inclination of the orbital plane of the Earth to the equatorial plane of the central body
i_E'	angle of inclination of the orbital plane of the Earth to its equatorial plane for 1950.0
i_E''	angle of inclination of the orbital plane of the Earth to the ecliptic plane for 1950.0
i_N	angle of inclination of the equatorial plane of the central body to the equatorial plane of the Earth for 1950.0
$\bar{i}$	unit vector along the x-axis
I_1, I_2	constants depending on the angle of inclination i
$\bar{j}$	unit vector along the y-axis
J_2	coefficient of the second harmonic of the potential of the central body
$\bar{k}$	unit vector along the z-axis
K_0	difference between the cosines of angle $\theta - 2\omega$ at time t and time t_0
K_N	difference between the cosines of angle $N\theta + 2\omega$ at time t and time t_0 for $N = 1, 2, \cdots, 5$
$K(a), K(i), K(\Omega), K(\omega)$	coefficients of expressions for the time variations of osculating orbital parameters

L_0	difference between the sines of angle $\theta - 2\omega$ at time t and time t_0
L_N	difference between the sines of angle $N\theta + 2\omega$ at time t and time t_0 for $N = 1, 2, \cdots, 5$
M	mass of the central body
M(t)	mean anomaly angle of the particle in its true orbit at time t
$M^*(t)$	mean anomaly angle of an imaginary particle moving in the initial osculating reference orbit at time t
n	mean motion of the particle in its orbit
$n_0 = n(0)$	mean motion of the particle at time t_0
N	subscript
p	semi-latus rectum of the osculating ellipse at the time t
$p_0 = p(0)$	semi-latus rectum of the initial osculating reference ellipse
$P_n(\zeta)$	Legendre polynomial of order n
$P_2(\zeta)$	Legendre polynomial of order two
r	magnitude of the heliocentric position vector of the spacecraft
r_0	magnitude of the heliocentric position vector of the spacecraft at time t_0
r_E	magnitude of the heliocentric position vector of the Earth
r^*	magnitude of the heliocentric position vector of an imaginary particle moving in the initial osculating reference ellipse at time t
$\dot{r}$	radial velocity of the spacecraft
$\dot{r}_E$	radial velocity of the Earth
$\overline{r}$	heliocentric position vector of the spacecraft at time t
$\overline{r}_0$	heliocentric position vector of the spacecraft at time t_0

$\bar{r}^*$	heliocentric position vector of an imaginary particle moving in the initial osculating reference ellipse at time t
$\bar{r}_E$	heliocentric position vector of the Earth at time t
$\bar{r}_E^*$	heliocentric position vector of an imaginary Earth moving in an initial osculating reference ellipse without perturbations
$\bar{\dot{r}}$	velocity vector of the spacecraft
$\bar{\dot{r}}$	velocity vector of the Earth
$\bar{\dot{r}}_E^*$	velocity vector of an imaginary particle moving in the initial osculating ellipse without perturbations
$\bar{\dot{r}}_E^*$	velocity vector of an imaginary Earth moving in the undisturbed initial osculating ellipse
R_0	equatorial radius of the central body
S_N	difference between sines of the angle $N\theta$ at time t and initial time t_0 for N = 1, 2, 3
t	time
t_0	initial time of osculation (epoch)
T_0	time of periapsis passage
$u = \theta + \omega$	argument of latitude
u_E	argument of latitude of the Earth
$\dot{u}$	angular velocity of the spacecraft in its orbit
$\dot{u}_E$	angular velocity of the Earth in its orbit
v	magnitude of the velocity vector of the spacecraft
v_0	magnitude of the velocity vector of the spacecraft at time t_0
$\bar{v}$	velocity vector of the spacecraft
$\bar{v}_E$	velocity vector of the Earth
V	potential of the central body

$V(0)$	potential of the central body at time t_0
x, y, z	heliocentric rectangular coordinates of the spacecraft in the equatorial reference frame of the central body
$\dot{x}, \dot{y}, \dot{z}$	velocity components of the spacecraft along the axes of the equatorial reference frame of the central body
x_E, y_E, z_E	heliocentric rectangular coordinates of the Earth in the equatorial reference frame of the central body
$\dot{x}_E, \dot{y}_E, \dot{z}_E$	velocity components of the Earth along the axes of the equatorial reference frame of the central body
X, Y, Z	heliocentric rectangular coordinates of the spacecraft in the equatorial reference frame of the Earth
$\dot{X}, \dot{Y}, \dot{Z}$	velocity components of the spacecraft along the axes of the equatorial reference frame of the Earth
X_E, Y_E, Z_E	heliocentric rectangular coordinates of the Earth in the equatorial reference frame of the Earth
$\dot{X}_E, \dot{Y}_E, \dot{Z}_E$	velocity components of the Earth along the axes of the equatorial reference frame of the Earth
α	right ascension of the north pole of the central body
Υ	vernal equinox point for 1950.0
δ	declination of the north pole of the central body
ϵ	a constant
ϵ_E	obliquity of the ecliptic
ζ	argument of Legendre polynomials; sine of the latitude angle
$\zeta_0 = \zeta(0)$	value of ζ at time t_0
θ	true anomaly angle of the spacecraft
θ^*	true anomaly angle of an imaginary particle moving in the unperturbed initial reference ellipse
θ_E	true anomaly angle of the Earth
θ	angular velocity of the spacecraft in its orbit

$\dot{\theta}_E$	angular velocity of the Earth in its orbit
θ_0	true anomaly of the spacecraft at time t_0
$\mu = GM$	gravitational constant of the central body
$\bar{\xi}$	geocentric position vector of the observer, $\xi = \lvert \bar{\xi} \rvert$
$\dot{\bar{\xi}}$	velocity vector of the observer
ρ	topocentric range of the spacecraft
$\dot{\rho}$	topocentric range rate: radial velocity of the spacecraft relative to the observer's position
$\bar{\rho}$	topocentric position vector of the spacecraft
$\dot{\bar{\rho}}$	velocity vector of the spacecraft relative to the observer
$\bar{\rho}*$	topocentric position vector of an imaginary particle moving in the unperturbed initial ellipse
$\dot{\bar{\rho}}*$	velocity vector, relative to the observer, of an imaginary particle moving in the initial unperturbed orbit
τ	difference between the mean anomalies of the spacecraft and an imaginary particle moving in the undisturbed orbit
ϕ	latitude (declination) of the spacecraft
ψ	eccentric anomaly angle of the spacecraft
ω	argument of perihelion of the spacecraft measured from the intersection of its orbital plane with the equatorial plane of the central body
ω_E	argument of perihelion of the Earth measured from the intersection of the Earth's orbital plane with the equatorial plane of the central body
ω'	argument of perihelion of the spacecraft measured from the intersection line of the Earth's orbital and the equatorial planes

ω'_E	argument of perihelion of the Earth measured from the intersection line of the Earth's orbital and equatorial planes for 1950.0
ω''_E	argument of perihelion of the Earth measured from the intersection of the Earth's orbital plane and the ecliptic plane for 1590.0
Ω	longitude of the ascending node of the spacecraft's orbital plane with respect to the equatorial plane of the central body
Ω_E	longitude of the ascending node of the Earth's orbital plane with respect to the equatorial plane of the central body
Ω'	longitude of the ascending node of the orbital plane of the spacecraft relative to the equatorial plane of the Earth for 1950.0
Ω'_E	longitude of the ascending node of the orbital plane of the Earth relative to its equatorial plane for 1950.0
Ω''_E	longitude of the ascending node of the orbital plane of the Earth relative to the ecliptic plane for 1950.0

REFERENCES

1. Georgevic, R.M., Trajectory Computation of a Vehicle Orbiting About a Rotationally Symmetric Primary, Technical Memorandum 391-106, Jet Propulsion Laboratory, Pasadena Calif., July 1, 1970 (JPL internal document).

2. Battin, R.H., Astronautical Guidance, McGraw Hill Book Co., New York, 1964.

3. Georgevic, R.M., Simplified Formulae for the Calculation of Perturbations and of the Range-Rate of a Celestial Body, Technical Memorandum 33-481, Jet Propulsion Laboratory, Pasadena, Calif., June 15, 1971.

4. Milankovic, M., Fundamentals of Celestial Mechanics, University of Belgrade, Yugoslavia, 1947.

5. Esposito, P. B., private communication, 1972.

6. Melbourne, W.G., et al., Constants and Related Information for Astrodynamic Calculations, 1968, Technical Report 32-1306, Jet Propulsion Laboratory, Pasadena, Calif., July 15, 1968.

7. Sturms, F.M., Jr., Polynomial Expressions for Planetary Equators and Orbit Elements With Respect to the Mean 1950.0 Coordinate System, Technical Report 32-1508, Jet Propulsion Laboratory, Pasadena, Calif., Jan. 15, 1971.

8. Georgevic, R.M., Analytic Expressions for Partial Derivatives of Range and Doppler With Respect to the Robertson's Relativistic Parameters, Technical Memorandum 391-88, Jet Propulsion Laboratory, Pasadena, Calif., April 24, 1970 (JPL internal document).

9. Georgevic, R.M., "Analytic Expressions for the Partial Derivatives of Observables With Respect to the Robertson's Relativistic Parameters," in Research and Advanced Development, Space Programs Summary 37-64, Vol. III, Jet Propulsion Laboratory, Pasadena, Calif., Aug. 31, 1970.

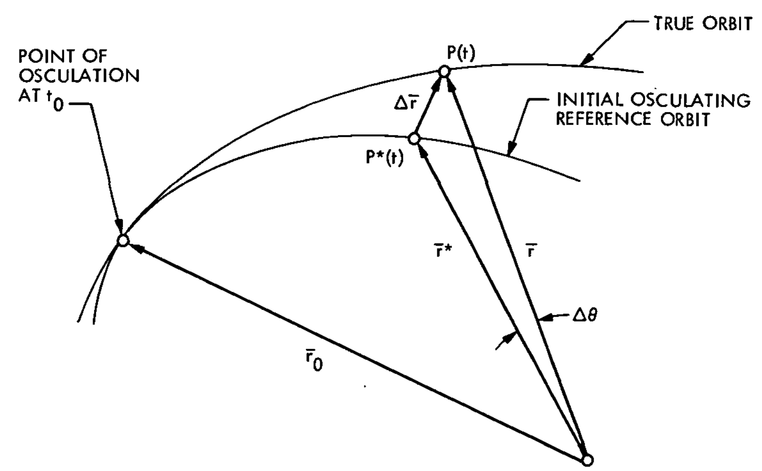

Fig. 1. Initial osculating orbit

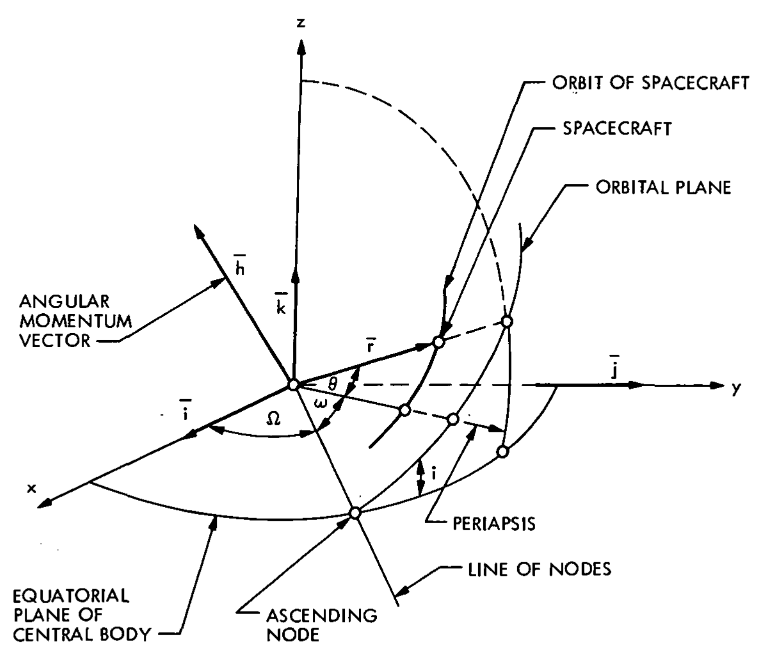

Fig. 2. Orbital geometry

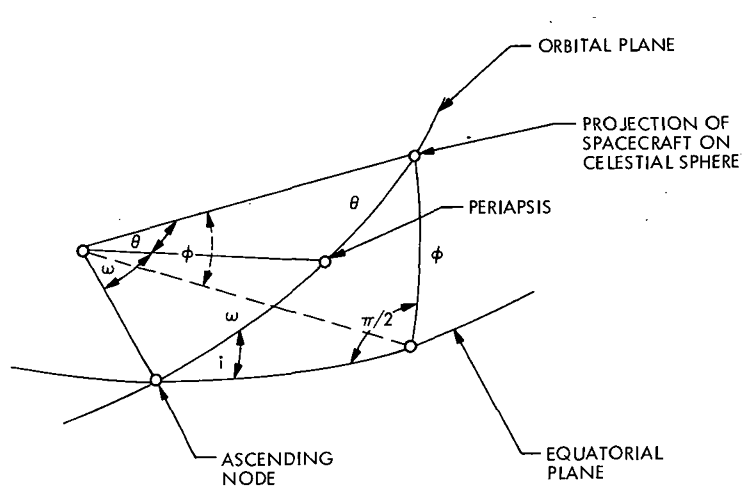

Fig. 3. Argument of latitude

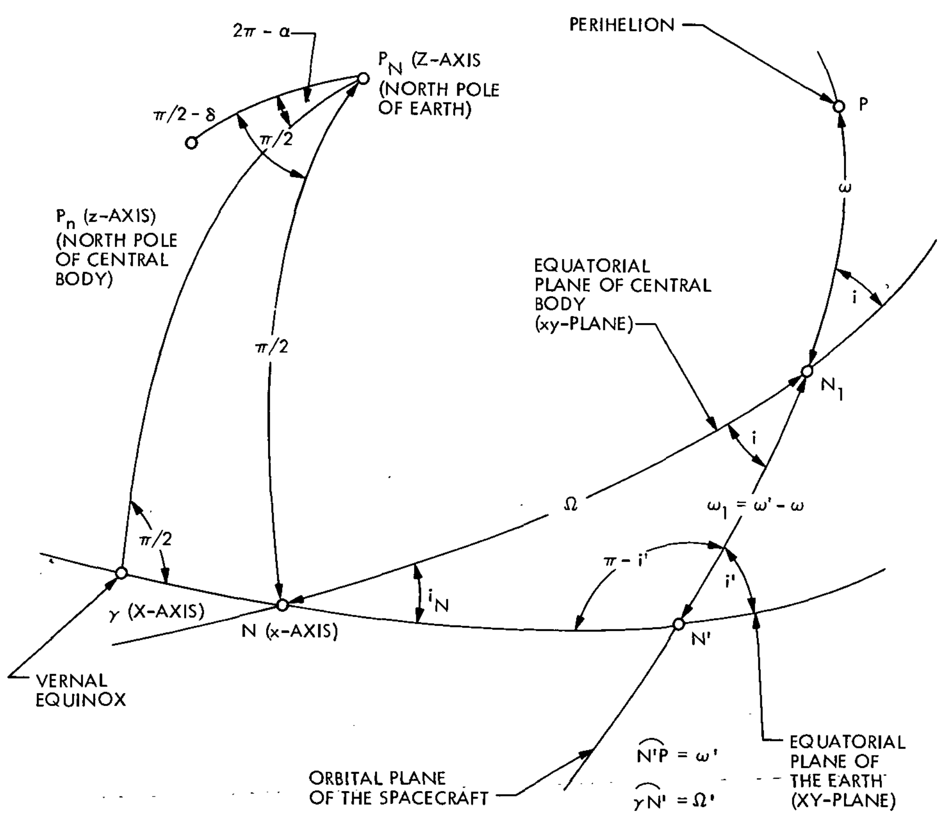

Fig. 4. Fundamental reference planes

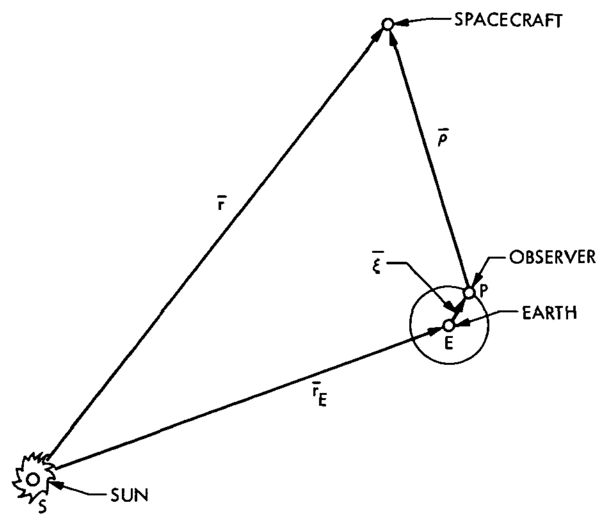

Fig. 5. Topocentric range of the spacecraft

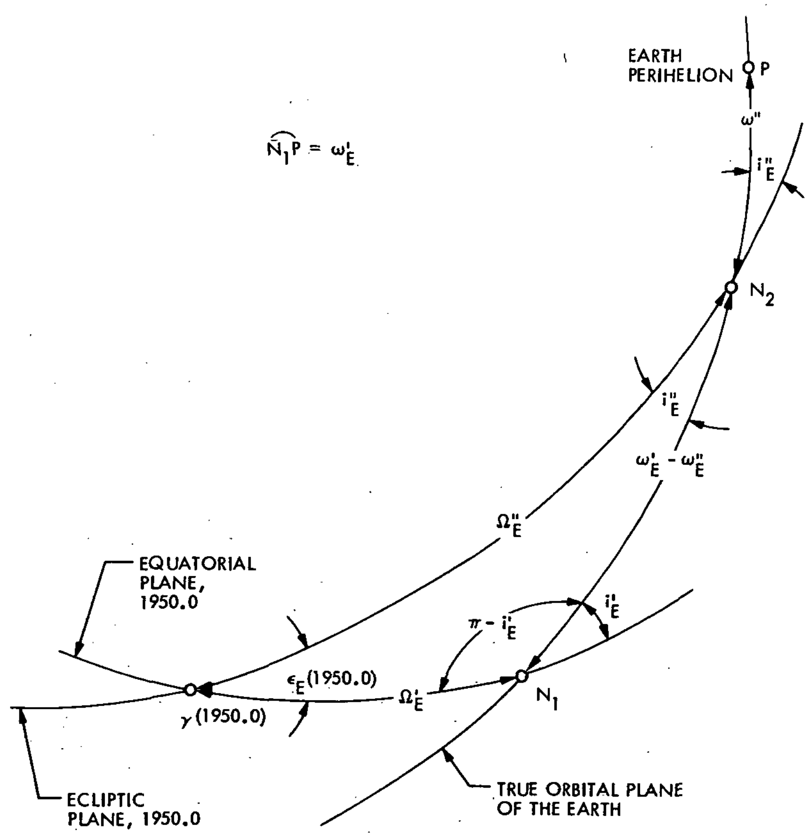

Fig. 6. Orbital geometry of the Earth

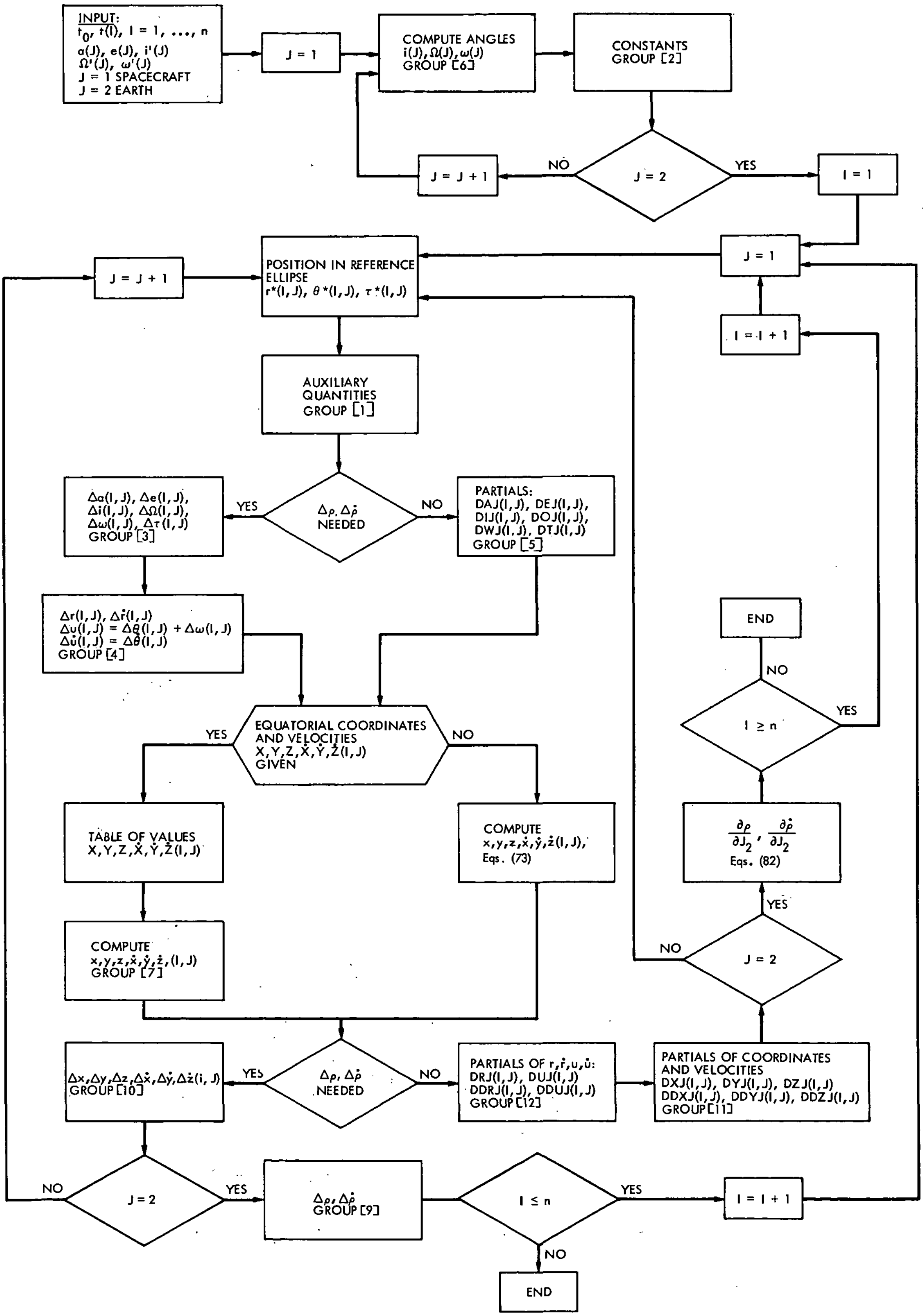

Fig. 7. Flow chart of the computational logic

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